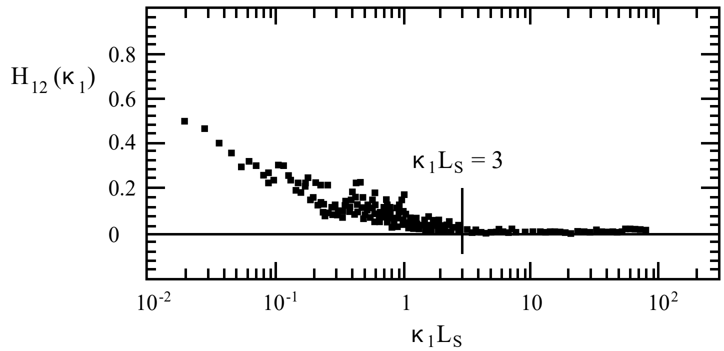

Figure 6.27: Spectral coherency measured in a turbulent boundary layer at $R_\lambda = 1400$ (Saddoughi and Veeravalli 1994).

$$u_{\mathcal{S}} \equiv (\varepsilon/\mathcal{S})^{\frac{1}{2}} = \alpha^{\frac{1}{2}} \left(\frac{\mathcal{P}}{\varepsilon}\right)^{-\frac{1}{2}} k^{\frac{1}{2}} \approx \tfrac{1}{2} k^{\frac{1}{2}}. \tag{6.283}$$

6.6 Spectral View of the Energy Cascade

In Sections 6.2–6.5 we introduced several statistics used to quantify turbulent motions on different scales, and we examined these statistics through experimental data, the Kolmogorov hypotheses, and a simple model spectrum. We are now in a position to provide a fuller account of the energy cascade than is given in Section 6.1. This Section therefore serves to summarize and consolidate the preceding development.

Energy-Containing Motions. We again consider very-high-Reynolds-number flow, so that there is a clear separation between the energy-containing and dissipative scales of motion (i.e., $L_{11}/\eta \sim \mathrm{Re}^{\frac{3}{4}} \gg 1$). The bulk of the turbulent kinetic energy is contained in motions of lengthscale ℓ, comparable to the integral lengthscale L_{11} ($6L_{11} > \ell > \frac{1}{6}L_{11} = \ell_{EI}$, say), and whose characteristic velocity is of order $k^{\frac{1}{2}}$. Since their size is comparable to the flow dimensions $\mathcal{L}$, these large-scale motions can be strongly influenced by the geometry of the flow. Furthermore, their timescale $L_{11}/k^{\frac{1}{2}}$ is large compared to the mean flow timescale (see Table 5.2 on page 135), so that they are significantly affected by the flow history. In other words, and in contrast to the universal equilibrium range, the energy-containing motions do not have a universal form brought about by a statistical equilibrium.

All of the anisotropy is confined to the energy-containing motions, and consequently so also is all of the turbulence production. On the other hand, the viscous dissipation is negligible. Instead, as the initial steps in the cascade, energy is removed by inviscid processes and transferred to smaller scales ($\ell < \ell_{EI}$) at a rate $\mathcal{T}_{EI}$, which scales as $k^{\frac{3}{2}}/L_{11}$. This transfer process depends on the non-universal energy-containing motions, and consequently the non-dimensional ratio $\mathcal{T}_{EI}/(k^{\frac{3}{2}}/L_{11})$ is not universal.

Energy Spectrum Balance. For homogeneous turbulence (with imposed mean velocity gradients) this picture is quantified by the balance equation for the energy spectrum function $E(\kappa, t)$. This equation (derived in detail in Hinze 1975 and Monin and Yaglom 1975) can be written

$$\frac{\partial}{\partial t}E(\kappa, t) = \mathcal{P}_\kappa(\kappa, t) - \frac{\partial}{\partial \kappa}\mathcal{T}_\kappa(\kappa, t) - 2\nu\kappa^2 E(\kappa, t). \tag{6.284}$$

The three terms on the right-hand side represent production, spectral transfer and dissipation.

The production spectrum $\mathcal{P}_\kappa$ is given by the product of the mean velocity gradients $\partial\langle U_i\rangle/\partial x_j$ and an anisotropic part of the spectrum tensor. The contribution to the production from the wavenumber range (κ_a, κ_b) is denoted by

$$\mathcal{P}_{(\kappa_a,\kappa_b)} = \int_{\kappa_a}^{\kappa_b} \mathcal{P}_\kappa \,\mathrm{d}\kappa, \tag{6.285}$$

and to the extent that all of the anisotropy is contained in the energy-containing range, we therefore have

$$\mathcal{P} = \mathcal{P}_{(0,\infty)} \approx \mathcal{P}_{(0,\kappa_{EI})}, \tag{6.286}$$

and

$$\mathcal{P}_{(\kappa_{EI},\infty)}/\mathcal{P} \ll 1. \tag{6.287}$$

In the second term on the right-hand side of Eq. (6.284), $\mathcal{T}_\kappa(\kappa)$ is the *spectral energy transfer rate*: it is the net rate at which energy is transferred from modes of lower wavenumber than κ to those higher than κ. This is simply related to $\mathcal{T}(\ell)$—the rate of transfer of energy from eddies larger than ℓ to those smaller than ℓ—by

$$\mathcal{T}(\ell) = \mathcal{T}_\kappa(2\pi/\ell). \tag{6.288}$$

The rate of gain of energy in the wavenumber range (κ_a, κ_b) due to this spectral transfer is

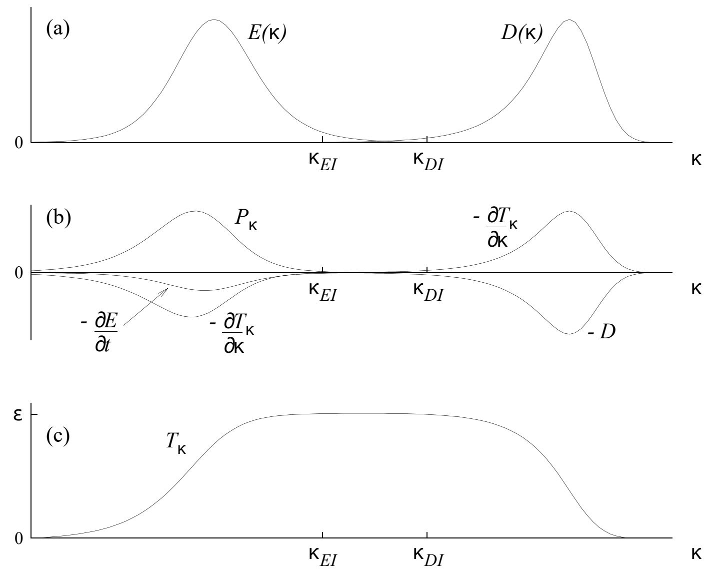

Figure 6.28: For homogeneous turbulence at very high Reynolds number, sketches of (a) the energy and dissipation spectra (b) the contributions to the balance equation for $E(\kappa, t)$ (Eq. 6.284), and (c) the spectral energy transfer rate.

$$\int_{\kappa_a}^{\kappa_b} -\frac{\partial}{\partial \kappa}\mathcal{T}_\kappa(\kappa)\,\mathrm{d}\kappa = \mathcal{T}_\kappa(\kappa_a) - \mathcal{T}_\kappa(\kappa_b). \tag{6.289}$$

Since $\mathcal{T}_\kappa$ vanishes at zero and infinite wavenumber, this transfer term makes no contribution to the balance of turbulent kinetic energy k.

An exact expression for $\mathcal{T}_\kappa$ can be obtained from the Navier-Stokes equations (see, e.g., Hinze 1975). There are two contributions: one resulting from interactions of triads of wavenumber modes, similar to Eq. (6.162); the other (examined in detail in Section 11.4) expressing a primarily kinematic effect that mean velocity gradients have on the spectrum.

The final term in Eq. (6.284) is the dissipation spectrum $D(\kappa, t) = 2\nu\kappa^2 E(\kappa, t)$.

Figure 6.28 is a sketch of the quantities appearing in the balance equation for $E(\kappa, t)$. In the energy-containing range, all the terms are significant except for dissipation. With the approximations $k_{(0,\kappa_{EI})} \approx k, \varepsilon_{(0,\kappa_{EI})} \approx 0$ and $\mathcal{P}_{(0,\kappa_{EI})} \approx \mathcal{P}$, when integrated over the energy-containing range $(0, \kappa_{EI})$, Eq. (6.284) yields

$$\frac{\mathrm{d}k}{\mathrm{d}t} \approx \mathcal{P} - \mathcal{T}_{EI}, \tag{6.290}$$

where $\mathcal{T}_{EI} = \mathcal{T}_\kappa(\kappa_{EI})$. In the inertial subrange, spectral transfer is the only significant process so that (when integrated from κ_{EI} to κ_{DI}) Eq. (6.284) yields

$$0 \approx \mathcal{T}_{EI} - \mathcal{T}_{DI}, \tag{6.291}$$

where $\mathcal{T}_{DI} = \mathcal{T}_\kappa(\kappa_{DI})$. While in the dissipation range, spectral transfer balances dissipation so that (when integrated from κ_{DI} to infinity) Eq. (6.284) yields

$$0 \approx \mathcal{T}_{DI} - \varepsilon. \tag{6.292}$$

When added together, the last three equations give (without approximation) the turbulent kinetic energy equation $\mathrm{d}k/\mathrm{d}t = \mathcal{P} - \varepsilon$.

The above equations again highlight the essential characteristics of the energy cascade. The rate of energy transfer from the energy-containing range $\mathcal{T}_{EI}$ depends, in a non-universal way, on several factors including the mean velocity gradients and the details of the energy-containing range of the spectrum. But this transfer rate then establishes an inertial subrange of universal character with $\mathcal{T}_\kappa(\kappa) = \mathcal{T}_{EI}$; and finally the high wavenumber part of the spectrum dissipates the energy at the same rate as it receives it. Thus both $\mathcal{T}_{DI}$ and ε are determined by, and are equal to, $\mathcal{T}_{EI}$. Quite often, when "dissipation" is being considered—e.g., in characterizing the inertial range spectrum as $E(\kappa) = C\varepsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}$—it is conceptually superior to consider $\mathcal{T}_{EI}$ in place of ε.

Cascade Timescale. An analogy of questionable validity is that the flow of energy in the inertial subrange is like the flow of an incompressible fluid through a variable-area duct. The constant flow rate is τ_{EI} (in units of energy per time) while the capacity of the cascade (analogous to duct area) is $E(\kappa)$ (in units of energy per wavenumber). So the speed (in units of wavenumber per time) at which the energy travels through the cascade is

$$\dot{\kappa}(\kappa) = \tau_{EI}/E(\kappa) = \kappa^{\frac{5}{3}}\varepsilon^{\frac{1}{3}}/C, \tag{6.293}$$

the latter expression being obtained from the Kolmogorov spectrum and the substitution $\tau_{EI} = \varepsilon$. Notice that this speed increases rapidly with increasing wavenumber.

It follows from the solution of the equation $\mathrm{d}\kappa/\mathrm{d}t = \dot{\kappa}$ that, according to this analogy, the time $t_{(\kappa_a,\kappa_b)}$ that it takes for energy to flow from wavenumber κ_a to the higher wavenumber κ_b is

$$\begin{aligned} t_{(\kappa_a,\kappa_b)} &= \tfrac{3}{2}C\varepsilon^{-\frac{1}{3}}\left(\kappa_a^{-\frac{2}{3}} - \kappa_b^{-\frac{2}{3}}\right) \\ &= \tau\tfrac{3}{2}C\left([\kappa_a L]^{-\frac{2}{3}} - [\kappa_b L]^{-\frac{2}{3}}\right). \end{aligned} \tag{6.294}$$

With the relations $\kappa_{EI} = 2\pi/\ell_{EI}, \ell_{EI} = \frac{1}{6}L_{11}$ and $L_{11}/L \approx 0.4$, this formula yields

$$t_{(\kappa_{EI},\infty)} \approx \tfrac{1}{10}\tau, \tag{6.295}$$

giving the estimate that the lifetime of the energy once it enters the inertial subrange is just a tenth of its total lifetime $\tau = k/\varepsilon$.

Spectral Energy Transfer Models. In the universal equilibrium range ($\kappa > \kappa_{EI}$), the balance in the spectral energy equation (Eq. 6.284) is between the energy transfer and the dissipation, see Fig. 6.28(b). Hence, (at any time t) Eq. (6.284) reduces to

$$0 = -\frac{\mathrm{d}}{\mathrm{d}\kappa}\mathcal{T}_\kappa(\kappa) - 2\nu\kappa^2 E(\kappa). \tag{6.296}$$

In the period from 1940 to 1970 many models were proposed for the spectral energy transfer rate $\mathcal{T}_\kappa$, which allow the form of the spectrum $E(\kappa)$ to be deduced from Eq. (6.296). The proposals of Obukhov (1941), Heisenberg (1948) and many others are reviewed by Panchev (1971). Appropriate to the physics of the cascade, most of these models are non-local in the sense that $\mathcal{T}_\kappa(\kappa)$ is postulated to depend on $E(\kappa')$, for $\kappa' \neq \kappa$. However, to illustrate the approach, we consider the simple local model due to Pao (1965) . Similar to Eq. (6.293), the speed of energy transfer $\dot{\kappa}(\kappa)$ is defined by

$$\dot{\kappa}(\kappa) \equiv \mathcal{T}_\kappa(\kappa)/E(\kappa). \tag{6.297}$$

The single (though strong) assumption in Pao's model is that $\dot{\kappa}$ depends solely on ε and κ. Dimensional analysis then determines

$$\mathcal{T}_\kappa(\kappa) = E(\kappa)\dot{\kappa}(\kappa) = E(\kappa)\alpha^{-1}\varepsilon^{\frac{1}{3}}\kappa^{\frac{5}{3}}, \tag{6.298}$$

where α is a constant. With this expression for $\mathcal{T}_\kappa$, Eq. (6.296) can be integrated (see Exercise 6.36) to yield the Pao spectrum

$$E(\kappa) = C\varepsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}\exp\left(-\tfrac{3}{2}C[\kappa\eta]^{\frac{4}{3}}\right), \tag{6.299}$$

cf. Eq. (6.254). This is compared to experimental data in Fig. 6.15.

Exercise 6.36 Substitute Eq. (6.298) into Eq. (6.296) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}\ln\left[E(\kappa)\kappa^{\frac{5}{3}}\right] = -2\alpha\nu\varepsilon^{-\frac{1}{3}}\kappa^{\frac{1}{3}}, \tag{6.300}$$

and then integrate to obtain

$$\begin{aligned} E(\kappa) &= \beta\kappa^{-\frac{5}{3}} \exp\left(-\tfrac{3}{2}\alpha\nu\varepsilon^{-\frac{1}{3}}\kappa^{\frac{4}{3}}\right), \\ &= \beta\kappa^{-\frac{5}{3}} \exp\left(-\tfrac{3}{2}\alpha[\kappa\eta]^{\frac{4}{3}}\right), \end{aligned} \tag{6.301}$$

where β is a (dimensional) constant of integration. Argue that, for consistency with the Kolmogorov spectrum (for small $\kappa\eta$), β is required to be $\beta = C\varepsilon^{\frac{2}{3}}$. Show that the dissipation given by Eq. (6.301) is

$$\int_0^\infty 2\nu\kappa^2 E(\kappa)\,\mathrm{d}\kappa = \varepsilon^{\frac{1}{3}}\beta/\alpha, \tag{6.302}$$

and hence that α is identical to the Kolmogorov constant C. Confirm that, with $\beta = C\varepsilon^{\frac{2}{3}}$ and $\alpha = C$, Eq. (6.301) yields the Pao spectrum, Eq. (6.299).

6.7 Limitations, Shortcomings and Refinements

In considerations of turbulent motions of different scales, the notions of the energy cascade, vortex stretching, and the Kolmogorov hypotheses provide an invaluable conceptual framework. However, both conceptually and empirically, there are some shortcomings. Indeed since around 1960, a major line of research (theoretical, experimental and computational) has been to examine these shortcomings and to attempt to improve on the Kolmogorov hypotheses. While it is appropriate to provide some discussion of these issues here, it should be appreciated that they have minor impact on the study and modelling of turbulent flows. This is simply because the small scales ($\ell < \ell_{EI}$) contain little energy (and less anisotropy) and so have little direct effect on the flow.

6.7.1 Reynolds Number

A limitation of the Kolmogorov hypotheses is that they apply only to high-Reynolds-number flows, and that a criterion for "sufficiently high Reynolds number" is not provided. Many laboratory and practical flows have reasonably high Reynolds number (e.g., $\mathrm{Re} \approx 10,000, \mathrm{R}_\lambda \approx 150$), and yet even the motions on the dissipative scales are found to be anisotropic (see, e.g., George and Hussein 1991).